Name ____________________ Class ____________ Date ________

3.2 Complex Numbers

Essential Question: What is a complex number, and how can you add, subtract, and multiply complex numbers?

Resource Locker

Explore Exploring Operations Involving Complex Numbers

In this lesson, you'll learn to perform operations with *complex numbers*, which have a form similar to linear binomials such as $3 + 4x$ and $2 - x$.

Ⓐ Add the binomials $3 + 4x$ and $2 - x$.

Group like terms. $$(3 + 4x) + (2 - x) = \left(3 + \boxed{}\right) + \left(4x + \boxed{}\right)$$

Combine like terms. $$= \left(\boxed{} + \boxed{}\right)$$

Ⓑ Subtract $2 - x$ from $3 + 4x$.

Rewrite as addition. $$(3 + 4x) - (2 - x) = (3 + 4x) + \left(-2 + \boxed{}\right)$$

Group like terms. $$= \left(3 + \boxed{}\right) + \left(4x + \boxed{}\right)$$

Combine like terms. $$= \left(\boxed{} + \boxed{}\right)$$

Ⓒ Multiply the binomials $3 + 4x$ and $2 - x$.

Use FOIL. $$(3 + 4x)(2 - x) = 6 + (-3x) + \boxed{} + \boxed{}$$

Combine like terms. $$= 6 + \boxed{} + \boxed{}$$

Reflect

1. In Step A, you found that $(3 + 4x) + (2 - x) = 5 + 3x$. Suppose $x = i$ (the imaginary unit).

What equation do you get? ____________________

2. In Step B, you found that $(3 + 4x) + (2 - x) = 1 + 5x$. Suppose $x = i$ (the imaginary unit).

What equation do you get? ____________________

3. In Step C, you found that $(3 + 4x)(2 - x) = 6 + 5x - 4x^2$. Suppose $x = i$ (the imaginary unit). What equation do you get? How can you further simplify the right side of this equation?

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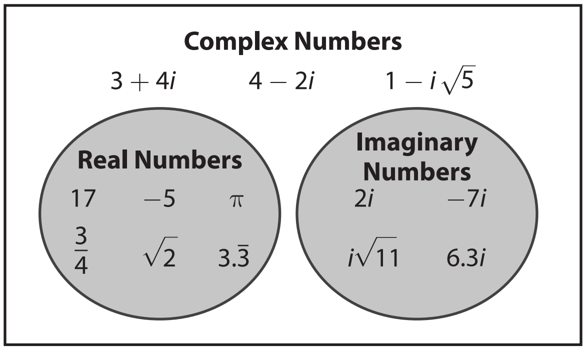
Explain 1 Defining Complex Numbers

A **complex number** is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. For a complex number $a + bi$ a is called the *real part* of the number, and b is called the *imaginary part*. (Note that "imaginary part" refers to the real multiplier of i; it does not refer to the imaginary number bi.) The Venn diagram shows some examples of complex numbers.



Notice that the set of real numbers is a subset of the set of complex numbers. That's because a real number a can be written in the form $a + 0i$ (whose imaginary part is 0). Likewise, the set of imaginary numbers is also a subset of the set of complex numbers, because an imaginary number bi (where $b \neq 0$) can be written in the form $0 + bi$ (whose real part is 0).

Example 1 **Identify the real and imaginary parts of the given number. Then tell which of the following sets the number belongs to: real numbers, imaginary numbers, and complex numbers.**

(A) $9 + 5i$

The real part of $9 + 5i$ is 9, and the imaginary part is 5. Because both the real and imaginary parts of $9 + 5i$ are nonzero, the number belongs only to the set of complex numbers.

(B) $-7i$

The real part of $-7i$ is ____, and the imaginary part is ____. Because the real/imaginary part is 0,

the number belongs to these sets: ______________________________.

Your Turn

Identify the real and imaginary parts of the given number. Then tell which of the following sets the number belongs to: real numbers, imaginary numbers, and complex numbers.

4. 11

5. $-1 + i$

Explain 2 Adding and Subtracting Complex Numbers

To add or subtract complex numbers, add or subtract the real parts and the imaginary parts separately.

Example 2 **Add or subtract the complex numbers.**

(A) $(-7 + 2i) + (5 - 11i)$

Group like terms. $(-7 + 2i) + (5 - 11i) = (-7 + 5) + (2i + (-11i))$

Combine like terms. $= -2 + (-9i)$

Write addition as subtraction. $= -2 - 9i$

(B) $(18 + 27i) - (2 + 3i)$

Group like terms. $(18 + 27i) - (2 + 3i) = \left(18 - \boxed{}\right) + \left(\boxed{} - 3i\right)$

Combine like terms. $= \boxed{} + \boxed{}\, i$

Reflect

6. Is the sum $(a + bi) + (a - bi)$ where a and b are real numbers, a real number or an imaginary number? Explain.

Your Turn

Add or subtract the complex numbers.

7. $(17 - 6i) - (9 + 10i)$

8. $(16 + 17i) + (-8 - 12i)$

Explain 3 Multiplying Complex Numbers

To multiply two complex numbers, use the distributive property to multiply each part of one number by each part of the other. Use the fact that $i^2 = -1$ to simplify the result.

Example 3 **Multiply the complex numbers.**

Ⓐ $(4 + 9i)(6 - 2i)$

Use the distributive property.	$(4 + 9i)(6 - 2i) = 24 - 8i + 54i - 18i^2$
Substitute -1 for i^2.	$= 24 - 8i + 54i - 18(-1)$
Combine like terms.	$= 42 + 46i$

Ⓑ $(-3 + 12i)(7 + 4i)$

Use the distributive property.	$(-3 + 12i)(7 + 4i) = \boxed{} - 12i + \boxed{} + 48i^2$
Substitute -1 for i^2.	$= \boxed{} - 12i + \boxed{} + 48(-1)$
Combine like terms.	$= \boxed{} + \boxed{}\,i$

Reflect

9. Is the product of $(a + bi)(a - bi)$, where a and b are real numbers, a real number or an imaginary number? Explain.

Your Turn

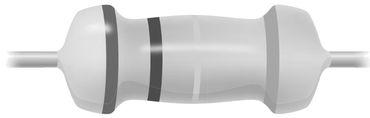
Multiply the complex numbers.

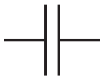
10. $(6 - 5i)(3 - 10i)$

11. $(8 + 15i)(11 + i)$

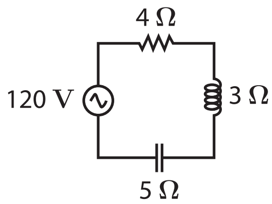
Explain 4 Solving a Real-World Problem Using Complex Numbers

Electrical engineers use complex numbers when analyzing electric circuits. An electric circuit can contain three types of components: resistors, inductors, and capacitors. As shown in the table, each type of component has a different symbol in a circuit diagram, and each is represented by a different type of complex number based on the phase angle of the current passing through it.



Circuit Component	Symbol in Circuit Diagram	Phase Angle	Representation as a Complex Number
Resistor		0°	A real number *a*
Inductor		90°	An imaginary number bi where $b > 0$
Capacitor		−90°	An imaginary number bi where $b < 0$

A diagram of an alternating current (AC) electric circuit is shown along with the *impedance* (measured in ohms, Ω) of each component in the circuit. An AC power source, which is shown on the left in the diagram and labeled 120 V (for volts), causes electrons to flow through the circuit. Impedance is a measure of each component's opposition to the electron flow.



Example 4 **Use the diagram of the electric circuit to answer the following questions.**

Ⓐ The total impedance in the circuit is the sum of the impedances for the individual components. What is the total impedance for the given circuit?

Write the impedance for each component as a complex number.

- Impedance for the resistor: 4
- Impedance for the inductor: $3i$
- Impedance for the capacitor: $-5i$

Then find the sum of the impedances.

Total impedance $= 4 + 3i + (-5i) = 4 - 2i$

Ⓑ Ohm's law for AC electric circuits says that the voltage V (measured in volts) is the product of the current I (measured in amps) and the impedance Z (measured in ohms): $V = I \cdot Z$. For the given circuit, the current I is $24 + 12i$ amps. What is the voltage V for each component in the circuit?

Use Ohm's law, $V = I \cdot Z$, to find the voltage for each component. Remember that Z is the impedance from Part A.

Voltage for the resistor $= I \cdot Z = (24 + 12i)\left(\boxed{}\right) = 96 + \boxed{}\,i$

Voltage for the inductor $= I \cdot Z = (24 + 12i)\left(\boxed{}\right) = -36 + \boxed{}\,i$

Voltage for the capacitor $= I \cdot Z = (24 + 12i)\left(\boxed{}\right) = \boxed{} - 120i$

Reflect

12. Find the sum of the voltages for the three components in Part B. What do you notice?

Your Turn

13. Suppose the circuit analyzed in Example 4 has a second resistor with an impedance of 2 Ω added to it. Find the total impedance. Given that the circuit now has a current of $18 + 6i$ amps, also find the voltage for each component in the circuit.

Elaborate

14. What kind of number is the sum, difference, or product of two complex numbers?

15. When is the sum of two complex numbers a real number? When is the sum of two complex numbers an imaginary number?

16. **Discussion** What are the similarities and differences between multiplying two complex numbers and multiplying two binomial linear expressions in the same variable?

17. **Essential Question Check-In** How do you add and subtract complex numbers?

Evaluate: Homework and Practice

- Online Homework
- Hints and Help
- Extra Practice

1. Find the sum of the binomials $3 + 2x$ and $4 - 5x$. Explain how you can use the result to find the sum of the complex numbers $3 + 2i$ and $4 - 5i$.

2. Find the product of the binomials $1 - 3x$ and $2 + x$. Explain how you can use the result to find the product of the complex numbers $1 - 3i$ and $2 + i$.

Identify the real and imaginary parts of the given number. Then tell which of the following sets the number belongs to: real numbers, imaginary numbers, and complex numbers.

3. $5 + i$

4. $7 - 6i$

5. 25

6. $i\sqrt{21}$

Add.

7. $(3 + 4i) + (7 + 11i)$

8. $(2 + 3i) + (6 - 5i)$

9. $(-1 - i) + (-10 + 3i)$

10. $(-9 - 7i) + (6 + 5i)$

Subtract.

11. $(2 + 3i) - (7 + 6i)$

12. $(4 + 5i) - (14 - i)$

13. $(-8 - 3i) - (-9 - 5i)$

14. $(5 + 2i) - (5 - 2i)$

Multiply.

15. $(2 + 3i)(3 + 5i)$

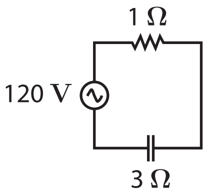
16. $(7 + i)(6 - 9i)$

17. $(-4 + 11i)(-5 - 8i)$

18. $(4 - i)(4 + i)$

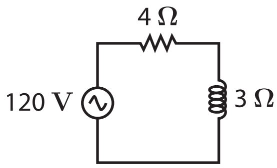
Use the diagram of the electric circuit and the given current to find the total impedance for the circuit and the voltage for each component.

19.



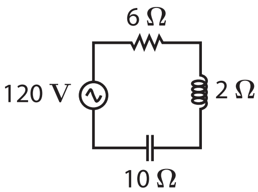
The circuit has a current of $12 + 36i$ amps.

20.



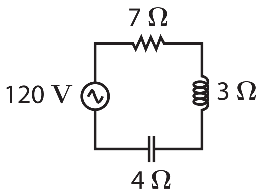
The circuit has a current of $19.2 - 14.4i$ amps.

21.



The circuit has a current of $7.2 + 9.6i$ amps.

22.



The circuit has a current of $16.8 + 2.4i$ amps.

23. Match each product on the right with the corresponding expression on the left.

A. $(3 - 5i)(3 + 5i)$ __________ $-16 + 30i$

B. $(3 + 5i)(3 + 5i)$ __________ -34

C. $(-3 - 5i)(3 + 5i)$ __________ 34

D. $(3 - 5i)(-3 - 5i)$ __________ $16 - 30i$

H.O.T. Focus on Higher Order Thinking

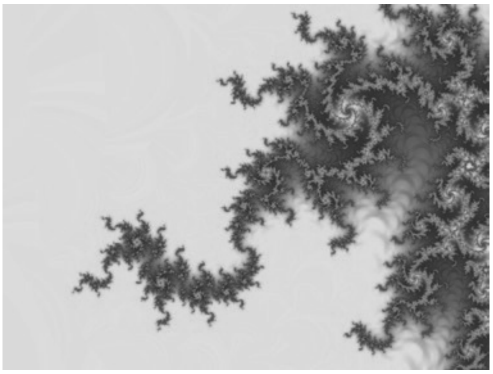
24. Explain the Error While attempting to multiply the expression $(2 - 3i)(3 + 2i)$, a student made a mistake. Explain and correct the error.

$$\begin{aligned}(2 - 3i)(3 + 2i) &= 6 - 9i + 4i - 6i^2 \\ &= 6 - 9(-1) + 4(-1) - 6(1) \\ &= 6 + 9 - 4 - 6 \\ &= 5\end{aligned}$$

25. Critical Thinking Show that $\sqrt{3} + i\sqrt{3}$ and $-\sqrt{3} - i\sqrt{3}$ are the square roots of $6i$.

26. Justify Reasoning What type of number is the product of two complex numbers that differ only in the sign of their imaginary parts? Prove your conjecture.

Lesson Performance Task



Just as real numbers can be graphed on a real number line, complex numbers can be graphed on a complex *plane*, which has a horizontal real axis and a vertical imaginary axis. When a set that involves complex numbers is graphed on a complex plane, the result can be an elaborate self-similar figure called a *fractal*. Such a set is called a Julia set.

Consider Julia sets having the quadratic recursive rule $f(n+1) = \left(f(n)\right)^2 + c$ for some complex number $f(0)$ and some complex constant c. For a given value of c, a complex number $f(0)$ either belongs or doesn't belong to the "'filled-in" Julia set corresponding to c depending on what happens with the sequence of numbers generated by the recursive rule.

a. Letting $c = i$, generate the first few numbers in the sequence defined by $f(0) = 1$ and $f(n+1) = \left(f(n)\right)^2 + i$. Record your results in the table.

n	$f(n)$	$f(n+1) = \left(f(n)\right)^2 + i$
0	$f(0) = 1$	$f(1) = \left(f(0)\right)^2 + i = (1)^2 + i = 1 + i$
1	$f(1) = 1 + i$	$f(2) = \left(f(1)\right)^2 + i = (1+i)^2 + i = \boxed{}$
2	$f(2) = \boxed{}$	$f(3) = \left(f(2)\right)^2 + i = \left(\boxed{}\right)^2 + i = \boxed{}$
3	$f(3) = \boxed{}$	$f(4) = \left(f(3)\right)^2 + i = \left(\boxed{}\right)^2 + i = \boxed{}$

b. The *magnitude* of a complex number $a + bi$ is the real number $\sqrt{a^2 + b^2}$. In the complex plane, the magnitude of a complex number is the number's distance from the origin. If the magnitudes of the numbers in the sequence generated by a Julia set's recursive rule, where $f(0)$ is the starting value, remain bounded, then $f(0)$ belongs to the "filled-in" Julia set. If the magnitudes increase without bound, then $f(0)$ doesn't belong to the "filled-in" Julia set. Based on your completed table for $f(0) = 1$, would you say that the number belongs to the "filled-in" Julia set corresponding to $c = i$? Explain.

c. Would you say that $f(0) = i$ belongs to the "filled-in" Julia set corresponding to $c = i$? Explain.